Stochastic Representation of Deterministic Interactions and Brownian Motion

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Exact generalized stochastic representation of deterministic interaction between two dynamical (quantum or classical) systems is derived which helps when considering one of them to replace another by equivalent commutative (*c*-number valued) random sources. The method is applied to classical Brownian motion of a particle in a gas, and statistics of this motion is reduced to statistics of the gas response to perturbations.

I. INTRODUCTION

In recent years the old variety of problems about interaction between one or another "dynamical system" (DS) and its environment (named also "thermal bath" or "thermostat") have attracted new attention as a part of the quantum computation problem. Frequently, the environment contains too many routine degrees of freedom to be of special interest in itself, and one would be satisfied by a simplified statistical description of DS in terms of its own degrees of freedom only but with adding of random sources, or "Langevin forces", which effectively substitute the "thermostat". In this respect, during last century many different approaches were proposed. Perhaps, the two extreme points among them are the highly time-nonlocal description of the DS evolution by means of path integrals and influence functionals [1], on one hand, and the mathematical theory of quantum semigroup evolution operators corresponding to purely time-local kinetics [2], on opposite hand.

In our previous works [3, 4, 5] the new intermediate approach was suggested, so-called "stochastic representation of quantum interactions". Its peculiarity is that it involves time-local stochastic evolution equation for density matrix of DS, with commutative (formally *c*-number valued) random sources which generally undergo time-nonlocal correlations. In contrary to phenomenological "Langevin equations" for DS variables, this equation is exact, in the sense that quite definite and transparent relations are established between complete statistics of the random sources and private statistical characteristics of the thermostat perturbed by classical (*c*-number valued) forces. These relations ensure adequate construction of approximate models of the thermostat noise in its connection with the thermostat induced dissipation.

In the present paper, most general formulation of this approach is presented which unifies both quantum and classical mechanics, at both Hamiltonian and non-Hamiltonian interactions between DS and thermostat.

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The results are illustrated by Brownian motion of classical particle in classical gas, the "trivial" phenomemon which still remains far from complete consideration (as well as Brownian motion of gas molecules themselves, see [7] and references therein).

II. STOCHASTIC REPRESENTATION OF TWO-SIDE INTERACTION

Let $\rho(t)$ be joint statistical operator (distribution function, density matrix) of combined system "DS plus thermostat (environment)" and $L(t)$ its evolution operator, so that

$$d\rho(t)/dt = L(t)\rho(t) \qquad (1)$$

We suppose that $L(t)$ has the bilinear form as follows:

$$L(t) = L_d(t) + L_b + L_{int} , \;\; L_{int} = \sum_n \Lambda_n^d \Lambda_n^b , \qquad (2)$$

where operators $L_d(t)$ and L_b are responsible for autonomous evolutions of DS and thermostat ("bath"), respectively, while the pairs Λ_n^d and Λ_n^b for their interaction. The indexes "d" or "b" mark objects relating to DS or thermostat, respectively. Notice that all of L_d and Λ_n^d commute with all of L_b and Λ_n^b, since these two sets of operators act onto variables of two different systems.

In general, $L_d(t)$ contains a part which describes external observation of DS. For instance, if DS undergoes quantum Hamiltonian evolution, then

$$L_d(t)\rho = \sum_k v_k(t)J_k \circ \rho + i[\rho, H_d(t)]/\hbar , \qquad (3)$$

where $H_d(t)$ is Hamiltonian of DS; J_k are some operators belonging to DS and representing quantum variables under observation; $v_k(t)$ are test functions related to J_k; $[,]$ and $\circ$ mean commutator and Jordan symmetric product, respectively:

$$[A,B] \equiv AB - BA , \;\; A \circ B \equiv (AB + BA)/2$$

In case of classical Hamiltonian evolution, similarly,

$$L_d(t)\rho = \sum_k v_k(t)J_k\rho + \{\rho, H_d(t)\} , \qquad (4)$$

with $\{,\}$ being the Poisson bracket, $\{A, B\} \equiv \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$ (q and p are canonical coordinates and momenta of DS), and J_k some phase functions (i.e. functions of q and p) under observation. Then the trace, $\text{Tr}\,\rho$ (or, in classical variant, integral over total phase space), differs from unit and gives characteristic functional of the observables $J_k(t)$, that is generating functional for their multi-time correlators [3, 4, 5].

Remembering that L_d and Λ_n^d commute with L_b and Λ_n^b, it is not hard to justify the formal identity

$$\overleftarrow{\exp}\left\{\int\left[L_d(t) + L_b + \sum \Lambda_n^d\Lambda_n^b\right]dt\right\} = \qquad (5)$$

$$= \overleftarrow{\exp}\left\{\int\left[L_b + \sum \Lambda_n^b\frac{\delta}{\delta\alpha_n(t)}\right]dt\right\}$$

$$\times \overleftarrow{\exp}\left\{\int\left[L_d(t) + \sum \alpha_n(t)\Lambda_n^d\right]dt\right\}\Big|_{\alpha(t)=0},$$

where $\overleftarrow{\exp}$ denotes chronologically ordered exponent. Because of the chronological ordering, this formula is the mere consequence from the identity $\exp(O_1 O_2) = \exp\left(O_1\frac{\partial}{\partial\alpha}\right)\exp(\alpha O_2)\big|_{\alpha=0}$, with O_1 and O_2 being two mutually commuting operators ($[O_1, O_2] = 0$).

From the other hand, if $\alpha_n(t)$ is a set of random processes with a given characteristic functional

$$\Psi\{f\} = \left\langle \exp\left[\int\sum f_n(t)\alpha_n(t)dt\right]\right\rangle$$

(the angle brackets mean statistical averaging), then the average of arbitrary functional $\Phi\{\alpha\}$ can be formally represented as

$$\langle\Phi\{\alpha\}\rangle = \Psi\left\{\frac{\delta}{\delta\alpha}\right\}\Phi\{\alpha\}\big|_{\alpha=0} \qquad (6)$$

Now, let us assume that more or less far in the past the joint statistical operator $\rho(t)$ was factorized: $\rho(t_0) = \rho_{d0}*\rho_{b0}$ (for instance, at $t_0 = -\infty$). Then comparison between the Eqs. 5 and 6 clearly results in two conclusions. First, the partial statistical operator of DS, $\overline{\rho}_d(t) \equiv \text{Tr}_b\,\rho(t)$, can be represented in the form

$$\overline{\rho}_d(t) \equiv \text{Tr}_b\,\rho(t) = \langle\rho_d(t)\rangle , \qquad (7)$$

where $\rho_d(t)$ is a solution (under initial condition $\rho_d(t_0) = \rho_{d0}$) to the stochastic evolution equation

$$d\rho_d(t)/dt = \left\{L_d(t) + \sum \alpha_n(t)\Lambda_n^d\right\}\rho_d(t) , \qquad (8)$$

with $\alpha_n(t)$ being random sources, and the angle brackets $\langle\rangle$ standing for the average with respect to these sources. Second, characteristic functional of these sources is defined by

$$\left\langle \exp\left[\int\sum f_n(t)\alpha_n(t)dt\right]\right\rangle = \qquad (9)$$

$$= \text{Tr}_b\,\overleftarrow{\exp}\left\{\int\left[L_b + \sum f_n(t)\Lambda_n^b\right]dt\right\}\rho_{b0}$$

Hence, from the point of view of DS its interaction with thermostat is equivalent to its separate evolution but disturbed by random sources and therefore described by stochastic statistical operator of DS, $\rho_d(t)$. The average $\langle\rho_d(t)\rangle$, being a functional of the test functions $v_k(t)$ introduced in (3) and (4), exactly reproduces not only current statistictical state of DS but also all the multi-time correlators of the observables J_k.

The Eqs.7-9 give what we call stochastic representation of interaction between two systems. In this representation, the DS and thermostat evolutions can be analysed separately. The cost of such possibility is that both DS and thermostat should be considered under arbitrary time-dependent perturbations ($\alpha_n(t)$ and $f_n(t)$, respectively). The examples [3, 4, 5] demonstrate that this cost may be not too large, since when considering DS we become saved of many degrees of freedom of thermostat and instead must deal with a few random sources only. Importantly, like c-numbers, the sources always commute one with another and with any other objects. Besides, possibly, their exact statistics which follows from (9), can be replaced by some simplified (semi-phenomenological) statistical model. Dependently on concrete contents of operators Λ_n^d and Λ_n^b, the sources $\alpha_n(t)$ behave either as literally classical random processes or as commutative "ghost fields" possessing unusual statistical properties (see [3, 4, 5] and below).

III. BILINEAR HAMILTONIAN INTERACTIONS

Let a quantum system "DS plus thermostat" has the Hamiltonian

$$H(t) = H_d(t) + H_b + H_{int} , \quad H_{int} = \sum_j B_j * D_j , \quad (10)$$

whose interaction part, H_{int}, is bilinear, with operators D_j and B_j acting in different Gilbert spaces of DS and thermostat, respectively, and therefore commuting one with another. The corresponding joint evolution operator is $L(t) = L_d(t) + L_b + L_{int}$, where $L_d(t)$ is given by (3), $L_b\rho = i[\rho, H_b]/\hbar$, and

$$L_{int}\rho = \sum \frac{i}{\hbar}[\rho, D_j * B_j] \qquad (11)$$

Because of $[D_j, B_j] = 0$, for any pair D_j and B_j the equalities take place:

$$[\rho, D * B] = [\rho, D]\circ B + D\circ[\rho, B] \qquad (12)$$

$$= [\rho\circ B, D] + [\rho\circ D, B]$$

Hence, the interaction part of the evolution operator has just the bilinear structure as in (2). We see that every

term $D_j * B_j$ from (10) produces two terms in L_{int} in (2), namely,

$$L_{int} = \sum_j \sum_{\sigma=1,2} \Lambda_{j\sigma}^d \Lambda_{j\sigma}^b \ , \qquad (13)$$

with the terms defined by

$$\Lambda_{j1}^d O \equiv i[O, D_j]/\hbar \, , \ \ \Lambda_{j1}^b O \equiv B_j \circ O \, , \qquad (14)$$

$$\Lambda_{j2}^d O \equiv D_j \circ O \, , \ \ \Lambda_{j2}^b O \equiv i[O, B_j]/\hbar$$

(here O is an arbitrary operator).

Correspondingly, every term of H_{int} results in two random sources in (8), $\alpha_{j\sigma}(t)$ ($\sigma = 1, 2$). It is convenient to rename them as $\alpha_{j1}(t) = x_j(t)$, $\alpha_{j2}(t) = y_j(t)$. After that the Eq.8 takes the form

$$\frac{d\rho_d}{dt} = L_d(t)\rho_d + \sum y_j(t) D_j \circ \rho_d + \sum x_j(t) \frac{i}{\hbar}[\rho_d, D_j] \tag{15}$$

According to (9) and (14), characteristic functional of the sources in this equation looks as

$$\left\langle \exp \int \sum [g_j(t)x_j(t) + f_j(t)y_j(t)]dt \right\rangle = \mathrm{Tr}_{\,b}\, \rho_b \ , \quad (16)$$

where operator ρ_b (effective partial statistical operator of thermostat) is defined as solution to the equation

$$\frac{d\rho_b}{dt} = L_b\rho_b + \sum g_j(t)B_j \circ \rho_b + \sum f_j(t) \frac{i}{\hbar}[\rho_b, B_j] \, , \quad (17)$$

with initial condition $\rho_b(t_0) = \rho_{b0}$. This equation describes separate evolution of thermostat, under its perturbation by classical (c-number valued) forces $f_j(t)$ conjugated with the observables B_j, and besides under measurement of the same observables. The $g_j(t)$ are the probe (test) functions of the measurement.

In the stochastic "Liouville-Langevin equation" (15), the sources $x(t)$ play the role of usual (realistic) random forces (potentials), while $y(t)$ are "ghost" ones. Indeed, if we put on $g(t) = 0$ in (17) then the evolution described by these equations becomes purely unitary. Therefore $\mathrm{Tr}_{\,b}\, \rho_b = 1$, and the Eq.16 yields $\left\langle \exp \int f(t)y(t)dt \right\rangle = 1$. Consequently, all the statistical moments of $y(t)$ and their correlations between themselves are zeros. Nevertheless, the joint statistical moments and cross correlations of $y(t)$ and $x(t)$ are nonzero, being responsible, in particular, for dissipation in DS [3, 4, 5]. According to (16) and (17),

$$\left\langle \prod_{j,m} x(t_j)y(\tau_m) \right\rangle = \left[\prod_m \frac{\delta}{\delta f(\tau_m)} \left\langle \prod_j B(t_j, f) \right\rangle \right]_{f=0} , \tag{18}$$

where $B(t, f)$ are the thermostat observables B considered as functions of time and functionals of the Hamiltonian perturbation characterized by the forces $f(t)$.

For details and another equivalent representations of the characteristic functional see [3, 4, 5]. It should be emphasized that due to the causality principle any of the correlators (18) turns into zero if $\max_m\{\tau_m\} > \max_j\{t_j\}$. In addition, without loss of generality, the operators B_j can be defined in such a way that $\langle x_j(t) \rangle = 0$.

In case of thermodynamically equilibrium thermostat, the important relations between pair correlators take place, namely,

$$\langle x_j(\tau)x_m(0) \rangle = \int_0^\infty \cos(\omega\tau) S_{jm}(\omega)\frac{d\omega}{\pi} \ ,$$

$$\langle x_j(\tau)y_m(0) \rangle = \vartheta(\tau) \times \qquad (19)$$

$$\times \frac{2}{\hbar} \int_0^\infty \sin(\omega\tau) \tanh\left(\frac{\hbar\omega}{2T}\right) S_{jm}(\omega)\frac{d\omega}{\pi}$$

Here T is the thermostat temperature, $\vartheta(\tau)$ is Heavyside function, and $S_{jm}(\omega)$ is a non-negatively defined spectral matrix. The connection between the two types of correlators in (19) merely expresses usual fluctuation-dissipation theorem. In accordance with (19), if we neglected the sources $y(t)$, it would be equivalent to infinite temperature.

IV. NON-HAMILTONIAN INTERACTIONS

We can obtain serious generalization of the representation (13)-(14) if replace (14) by

$$\Lambda_{j1}^d O \equiv i[O, D_j]/\hbar \, , \ \ \Lambda_{j1}^b O \equiv B_j' \circ O \, , \qquad (20)$$

$$\Lambda_{j2}^d O \equiv D_j' \circ O \, , \ \ \Lambda_{j2}^b O \equiv i[O, B_j]/\hbar \, ,$$

thus involving not two but four different observables at any j, with D' and D belonging to DS while B' and B to thermostat. Instead of (15) and (17), we come to the representation as follows (summation over j is taken in mind):

$$\frac{d\rho_d}{dt} = L_d(t)\rho_d + y(t) D' \circ \rho_d + x(t) \frac{i}{\hbar}[\rho_d, D] \, , \qquad (21)$$

$$\frac{d\rho_b}{dt} = L_b\rho_b + g(t)B' \circ \rho_b + f(t) \frac{i}{\hbar}[\rho_b, B] \qquad (22)$$

This is accompanied by the same definition (16) of the characteristic functional.

Obviously, the interaction defined by (13) and (20) is not reducible to a bilinear Hamiltonian interaction like (10), since any pair of the sources ($x(t)$ and $y(t)$) replaces not one but two different observables of the thermostat (B' and B). As in (21), $y(t)$ are "ghost" noise sources. The formula (18) extends to this non-Hamiltonian case under substituting variables $B'(t, f)$ in place of variables

$B(t, f)$ in (18). An exact contents of the correlators $\langle \prod B'(t_j, f) \rangle$ is directly implied by the definition (16).

The most general form of non-Hamiltonian interaction similar to (21) and (22) is expressed by the formulas

$$L_{int} = \sum \{\Lambda_j^d \Lambda_j^{b\prime} + \Lambda_j^{d\prime} \Lambda_j^b\} , \qquad (23)$$

$$\frac{d\rho_d}{dt} = L_d(t)\rho_d + \sum \{y_j(t)\Lambda_j^{d\prime} + x_j(t)\Lambda_j^d\}\rho_d , \qquad (24)$$

$$\frac{d\rho_b}{dt} = L_b\rho_b + \sum \{g_j(t)\Lambda_j^{b\prime} + f_j(t)\Lambda_j^b\}\rho_b , \qquad (25)$$

again accompanied by (16), where Λ_j^d and Λ_j^b are some generators of unitary (phase volume preserving) individual evolutions of DS and thermostat, respectively, while $\Lambda_j^{d\prime}$ and $\Lambda_j^{b\prime}$ generate non-unitary phase volume exchange between DS and thermostat (their mutual observation one for another).

V. BROWNIAN MOTION

Examples of the bilinear Hamiltonian interaction can be found in [3, 4, 5]. As concrete example of the previous non-Hamiltonian case, consider Brownian motion of classical particle in classical gas.

A. Brownian particle in a gas

Let $\boldsymbol{R}$, $\boldsymbol{V}$, $\boldsymbol{P}$ and M be position vector, velocity, momentum and mass, respectively, of Brownian particle (BP), $\boldsymbol{r}_j$, $\boldsymbol{v}_j$, $\boldsymbol{p}_j$ and m be analogous quantities of gas particles, and the latters interact with BP by mean of potentials $U(\boldsymbol{r}_j - \boldsymbol{R})$. The joint evolution operator is merely the Liouville operator of the system "DS and thermostat (gas)":

$$L = L_d(t) + L_g + \sum \boldsymbol{F}(\boldsymbol{r}_j - \boldsymbol{R}) \cdot \left(\frac{\partial}{\partial \boldsymbol{P}} - \frac{\partial}{\partial \boldsymbol{p}_j} \right) , \quad (26)$$

where $\boldsymbol{F}(\boldsymbol{r}) \equiv -\nabla U(\boldsymbol{r})$, and L_g is the Liouville operator of the gas itself. Generally L_d includes three terms:

$$L_d(t) = i\boldsymbol{k}(t) \cdot \boldsymbol{V} - \boldsymbol{V} \cdot \frac{\partial}{\partial \boldsymbol{R}} - \boldsymbol{F}_{ext}(t) \cdot \frac{\partial}{\partial \boldsymbol{P}} , \quad (27)$$

with $\boldsymbol{F}_{ext}(t)$ being an external force applied to BP, and $\boldsymbol{k}(t)$ the test (probe) function for observing BP's velocity ($i\boldsymbol{k}(t)$ plays the same role as $v(t)$'s in (4)).

Let us reduce expression (26) to the form of non-Hamiltonian bilinear interaction (2), (23). First, make the non-canonic change of variables, namely, consider positions of gas particles in terms of the relative distances $\boldsymbol{r}_j - \boldsymbol{R}$ and redenote the latters as $\boldsymbol{r}_j$. Under this change, L_g preserves its form, but $\boldsymbol{F}(\boldsymbol{r}_j - \boldsymbol{R})$ transforms into

$\boldsymbol{F}(\boldsymbol{r}_j)$, and $\frac{\partial}{\partial \boldsymbol{R}}$ transforms into $\frac{\partial}{\partial \boldsymbol{R}} - \sum \frac{\partial}{\partial \boldsymbol{r}_j}$. Second, introduce the operator

$$L_b = L_g - \sum \boldsymbol{F}(\boldsymbol{r}_j) \cdot \frac{\partial}{\partial \boldsymbol{p}_j} \qquad (28)$$

Formally, L_b describes the gas in presence of immovable scatterer fixed at zero point. After that (26) transforms into

$$L = L_d(t) + L_b + L_{int} , \qquad (29)$$

$$L_{int} = \sum \boldsymbol{F}(\boldsymbol{r}_j) \cdot \frac{\partial}{\partial \boldsymbol{P}} + \boldsymbol{V} \cdot \sum \frac{\partial}{\partial \boldsymbol{r}_j} \qquad (30)$$

Hence, L_{int} has just the structure (23), and we can identify the Λ's (replacing their indexes by natural vectorial notations):

$$\boldsymbol{\Lambda}^d = -\frac{\partial}{\partial \boldsymbol{P}} , \quad \boldsymbol{\Lambda}^{b\prime} = -\sum \boldsymbol{F}(\boldsymbol{r}_j) ,$$

$$\boldsymbol{\Lambda}^{d\prime} = \boldsymbol{V} = \frac{\boldsymbol{P}}{M} , \quad \boldsymbol{\Lambda}^b = \sum \frac{\partial}{\partial \boldsymbol{r}_j} \qquad (31)$$

Correspondingly, the Eqs. 24 and 25 look as

$$\frac{d\rho_d}{dt} = \left\{ L_d(t) + \boldsymbol{y}(t) \cdot \boldsymbol{V} - \boldsymbol{x}(t) \cdot \frac{\partial}{\partial \boldsymbol{P}} \right\} \rho_d , \qquad (32)$$

$$\frac{d\rho_b}{dt} = \left\{ L_b - \boldsymbol{g}(t) \cdot \sum \boldsymbol{F}(\boldsymbol{r}_j) + \boldsymbol{f}(t) \cdot \sum \frac{\partial}{\partial \boldsymbol{r}_j} \right\} \rho_b , \qquad (33)$$

to be accompanied by (27) and (28). For the characteristic functional (16) let us introduce the designation

$$\Xi\{\boldsymbol{g}, \boldsymbol{f}\} \equiv \left\langle e^{\int [\boldsymbol{g}(t) \cdot \boldsymbol{x}(t) + \boldsymbol{f}(t) \cdot \boldsymbol{y}(t)]dt} \right\rangle = \mathrm{Tr}\,_b\, \rho_b \qquad (34)$$

Clearly, in (32) $\boldsymbol{x}(t)$ is actual random Langevin force produced by the gas. In (33), factual physical dimensionality of the "force" $\boldsymbol{f}(t)$ is velocity.

Eventually we are interested in the characteristic functional of the velocity and displacement of BP, that is

$$\overline{\Theta}\{i\boldsymbol{k}, \boldsymbol{F}_{ext}\} \equiv \left\langle e^{\int i\boldsymbol{k}(t) \cdot \boldsymbol{V}(t)dt} \right\rangle \qquad (35)$$

It can be found in two steps:

$$\Theta\{i\boldsymbol{k}, \boldsymbol{F}_{ext}, x, y\} \equiv \int \rho_d \, dRdV , \qquad (36)$$

$$\overline{\Theta}\{i\boldsymbol{k}, \boldsymbol{F}_{ext}\} = \langle \Theta\{i\boldsymbol{k}, \boldsymbol{F}_{ext}, x, y\} \rangle \qquad (37)$$

The first step is very easy, due to that the coefficients of the stochastic Liouville operator on right-hand side of Eq.32 (first-order differential operator) do no depend on R and are linear functions of V. Supposing, without loss

of generality, that initially BP was fixed at zero point at its phase space, $\boldsymbol{R}(t_0) = 0$, $\boldsymbol{V}(t_0) = 0$, i.e. $\rho_{d0} = \delta(\boldsymbol{R})\delta(\boldsymbol{V})$, we can obtain

$$\Theta\{i\boldsymbol{k}, \boldsymbol{F}_{ext}, x, y\} = \qquad (38)$$

$$\exp\left\{\int_{t_2>t_1} [i\boldsymbol{k}(t_2) + \boldsymbol{y}(t_2)] \cdot [\boldsymbol{F}_{ext}(t_1) + \boldsymbol{x}(t_1)] \frac{dt_1 dt_2}{M}\right\}$$

In opposite, the second step, that is averaging (38) with respect to random trajectories $x(t)$ and $y(t)$, is rather nontrivial problem.

B. Path integral formulation

Of course, we can rearrange Φ and Ψ in the identity (6) and besides replace α_n by $c_n\alpha_n$ with c being any nonzero coefficients. Therefore let us write

$$\overline{\Theta}\{i\boldsymbol{k}, \boldsymbol{F}_{ext}\} = \left[\Theta\left\{i\boldsymbol{k}, \boldsymbol{F}_{ext}, \frac{\delta}{\delta ig}, \frac{\delta}{\delta f}\right\} \Xi\{ig, f\}\right]_{g=f=0}$$
$$(39)$$

Due to simple specific structure (38) of the functional Θ, we can easy transform the latter differential expression into integral one if use functional analogue of the formal identities

$$\exp\left(i\tau\frac{\partial^2}{\partial a \partial b}\right)\delta(a)\delta(b) = \frac{1}{2\pi\tau}\exp\left(\frac{iab}{\tau}\right), \qquad (40)$$

$$\left[\exp\left(i\tau\frac{\partial^2}{\partial a \partial b}\right)\Phi(a, b)\right]_{a=b=0} = \qquad (41)$$

$$= \frac{1}{2\pi\tau}\int da \int db \exp\left(\frac{iab}{\tau}\right)\Phi(a, b)$$

The final result of subsequent natural algebraic manipulations is the path integral

$$\overline{\Theta}\{i\boldsymbol{k}, \boldsymbol{F}_{ext}\} = \int DgDf \; \Xi\{ig, f\} \times \qquad (42)$$

$$\exp\left\{i\int\left[M\boldsymbol{f}(t)\frac{dg(t)}{dt} + \boldsymbol{k}(t)\boldsymbol{f}(t) + \boldsymbol{g}(t)\boldsymbol{F}_{ext}(t)\right]dt\right\},$$

with $DgDf \equiv \prod_t[Mdg(t)df(t)/2\pi]$ being path differential in some reasonable space of $g(t), f(t)$ trajectories. For brevity, here and below we omit the central dot symbolizing scalar product.

C. Limit of ideal gas

In the simplest case, let the gas be ideal, i.e. there are no interactions between its atoms and besides no initial statistical correlations between them:

$$L_g = -\sum_{k=1}^{N} \boldsymbol{v}_j \frac{\partial}{\partial \boldsymbol{r}_j}, \quad \rho_{b0} = \prod_{k=1}^{N} \frac{W_0(\boldsymbol{r}_k, \boldsymbol{v}_k)}{\Omega},$$

with Ω and N being total volume and number of atoms of gas, respectively. Naturally, the initial one-particle distribution function can be supposed homogeneous (space-independent at least far from BP), for instance, $W_0(\boldsymbol{r}, \boldsymbol{v}) = W_0(\boldsymbol{v})$, while $W_0(\boldsymbol{v})$ arbitrary even (spherically symmetric) velocity distribution. In particular, the reasonable choice for $W_0(\boldsymbol{v})$ is usual Maxwellian distribution corresponding to thermally equilibrium gas with a definite temperature.

In the "thermodynamic limit" $N \to \infty$, $\Omega \to \infty$, $N/\Omega \to \nu$ ($\nu =$const), statistics of the thermostat noise gets the general Poissonian form (i.e. becomes infinitely divisible):

$$\Xi\{\boldsymbol{g}, \boldsymbol{f}\} = \exp\left(\nu Q\{\boldsymbol{g}, \boldsymbol{f}\}\right) \qquad (43)$$

Here functional $Q\{\boldsymbol{g}, \boldsymbol{f}\}$ characterizes contribution to the overall noise from one atom:

$$Q\{\boldsymbol{g}, \boldsymbol{f}\} \equiv \int[W(t, \boldsymbol{r}, \boldsymbol{u}) - W_0(\boldsymbol{r}, \boldsymbol{u})]\,d\boldsymbol{r}d\boldsymbol{u},$$

with function $W(t, \boldsymbol{r}, \boldsymbol{v})$ undergoing the equation

$$\frac{dW}{dt} = \left\{-\boldsymbol{g}(t)\boldsymbol{F}(\boldsymbol{r}) + [\boldsymbol{f}(t) - \boldsymbol{v}]\frac{\partial}{\partial \boldsymbol{r}} - \frac{\boldsymbol{F}(\boldsymbol{r})}{m}\frac{\partial}{\partial \boldsymbol{v}}\right\}W$$
$$(44)$$

and the initial condition $W(t_0, \boldsymbol{r}, \boldsymbol{v}) = W_0(\boldsymbol{r}, \boldsymbol{v})$. Hence, analysis of Eqs. 33 and 34, in company with Eq.28, reduces to the one-particle problem.

In Eq.44 the first right-hand term says that an atom is observed by BP. The next two terms say that it is dynamically perturbed by BP, as if an atom would have the Hamiltonian $H = [\boldsymbol{p} - m\boldsymbol{f}(t)]^2/2m + U(\boldsymbol{r})$. Hence, the "force" $\boldsymbol{f}(t)$ representing velocity of BP acts like an effective time-varying vector potential.

Alternatively, we can consider this one-particle problem in terms of the factual velocity of an atom, $\boldsymbol{u} \equiv \boldsymbol{p}/m - \boldsymbol{f}(t)$, instead of its momentum $\boldsymbol{p}$ or $\boldsymbol{v} = \boldsymbol{p}/m$. Then Eq.44 transforms into

$$\frac{dW}{dt} = \left\{-\boldsymbol{g}(t)\boldsymbol{F}(\boldsymbol{r}) - \boldsymbol{u}\frac{\partial}{\partial \boldsymbol{r}} + \left[\frac{df(t)}{dt} - \frac{\boldsymbol{F}(\boldsymbol{r})}{m}\right]\frac{\partial}{\partial \boldsymbol{u}}\right\}W$$
$$(45)$$

In agreement with the above mentioned condition $\boldsymbol{V}(t_0) = 0$, we can assume that $\boldsymbol{f}(t_0) = 0$ and that initial condition for the W treated as a function of the factual velocity $\boldsymbol{u} = \boldsymbol{p}/m - \boldsymbol{f}(t)$ is $W(t_0, \boldsymbol{r}, \boldsymbol{u}) = W_0(\boldsymbol{r}, \boldsymbol{u})$. Because integration over $\boldsymbol{u}$ in (43) is indifferent to shift of the integration variable, $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{f}(t)$, the function $W(t, \boldsymbol{r}, \boldsymbol{u})$ in the characteristic functional (43) can be mentioned as solution to Eq.45.

VI. DISCUSSION AND RESUME

The above example once again demonstrates (see also [3, 4, 5]) how the stochastic representation formalism can help in reducing many-particle problems to one- or few-particle problems. It will be interesting to find out how

far this approach conducts in analyzing factual statistics of Brownian motion beyond conventional Gaussian approximation.

Of special interest is one-dimensional motion which arises in the problem about massive piston sliding in (infinitely long) tube filled with an ideal gas [6]. It was rigorously proved (see [6] for the review and original results) that in the limit of infinitely heavy BP (piston) its velocity tends expectedly to the Ornstein-Uhlenbeck random process possessing Gaussian statistics.

From the other hand, our (mathematically non-rigorous) attempt [7] (see also [8]) to reconsider derivation of Boltzmann kinetics of classical weakly non-ideal gas, in the Boltzmann-Grad limit, demonstrated that statistics of Brownian motion (self-diffusion) of gas atoms is essentially non-Gaussian because of low-frequency fluctuations in diffusivities of the atoms (uncertainty of the diffusivities).

Hence, the task naturally arises about non-Gaussian corrections to statistics of piston's motion at large but finite ratio M/m. In principle, it is sufficient to assume symmetrical situation when initially both "left-hand" and "right-hand" gases are in identical (mirror symmetrical) states. In order to keep constant piston's velocity relaxation time, the gas density ν (number of atoms per unit tube length) should be maintained proportionally to M/m.

Due to the symmetry, in Eq.43,

$$Q\{g, f\} = Q_+\{g, f\} + Q_+\{-g, -f\} \ ,$$

where $Q_+\{g, f\}$ is connected to solution of Eq.44 or Eq.45 for the right-hand gas atom by the same Eq.43. The source of possibly non-trivial corrections (as well as non-trivial mathematical difficulties, see [6]) is the recollisions between piston and atoms.

Indeed, in an infinitely long container (tube), let before a collision occuring at $t = 0$ the piston's velocity, $V(t)$, and a "right-hand" atom's velocity, $v(t)$, were equal to V_0 and v_0, respectively, of course, satisfying $v_0 < V_0$. After collision the atom's velocity equals to $v = 2V_0 - v_0$, while the distance, $\delta R(t)$, between piston and atom changes as

$$\delta R(t) = vt - R(t) \ , \ \ R(t) = \int_0^t V(t')dt'$$

The recollision will happen if $\delta R(t) = 0$ at $t > 0$. Suppose that after the collision the atom runs on the left, i.e. $v = 2V_0 - v_0 < 0$, which means that it was overtaking the piston, i.e. $V_0 < 0$. Besides let $V(t)$ is symmetric Ornstein-Uhlenbeck random process. Then, obviously, it is quite impossible that recollision will not happen, since this would mean that symmetric Brownian path $R(t)$ for the whole time $t > 0$ remains confined below the straight line vt with negative slope $v < 0$. In opposite, the recollision is inevitable. Consequently, its probability is not less than the probability that $2V_0 < v_0 < V_0 < 0$, that is (at nearly equilibrium situation) it has at least the order on $\sqrt{m/M}$.

Such estimates prompt that probabilities of multiple recollisions also are substantial. Due to all the recollisions, relatively slow atoms are accelerated by piston when it in its turn is accelerated by dominating relatively fast atoms. In other words, slow atoms "dress the piston in a coat" thus randomly changing its effective mass. Seemingly, this must imply some fluctuations in relaxation time and diffusivity of the piston. From the point of view of [7, 8] this effect associates with that space-drift velocity of probability of some collisions chain is the center-of-mass velocity of particles taking part in this chain. Whether low-frequency fluctuations in diffusivity of Brownian particle wandering in one-dimensional ideal gas really exist is the question for separate consideration.

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